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Determination of Subtraction Terms in S-Matrix Theory

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A detailed discussion of several points connected with the determination of subtraction constants in S-matrix theory is made. By considering examples of asymptotic behavior of scattering amplitudes characteristic of poles and cuts in the angular-momentum plane, explicit formulas for the subtraction functions in terms of the absorptive part are derived. It is emphasized that it is not necessary for all singularities to retreat to the left half angular-momentum plane in order for the subtraction functions to be completely determined. An explicit demonstration, for one example, is given of the fact that for nonpolynomial asymptotic behavior of scattering amplitudes in momentum transfer, the partial-wave amplitudes for all physical angular momenta are generally expected to be linked by analytic continuation in the complex angular-momentum plane. This property of partial-wave amplitudes is sometimes stated as an S-matrix postulate and referred to as maximal analyticity of the second degree.

ONE way of stating the problem of subtractions in dispersion relations is as follows: Suppose that the absorptive parts D_t and D_u in the Mandelstam amplitude are assumed to be given. Then to what extent is the amplitude in the s channel determined? (This way of formulating the problem has been utilized by Chew[1].) The answer to this question, which has become greatly clarified in recent years, depends upon the nature of the asymptotic behavior of the full amplitude, $A(s,t)$, and that of D_t and D_u in the t and u variables.

In potential scattering with a superposition of Yukawa potentials, the amplitude is uniquely determined by D_t and D_u, for in this case a knowledge of the absorptive parts is essentially equivalent to a knowledge of the potential (see Ref. 1).

In the relativistic S matrix, D_t and D_u can be computed in terms of mass-shell S-matrix elements and are often regarded as providing information analogous to the nonrelativistic potential.[2] We have, however, no guarantee in relativistic problems that undetermined subtractions may not enter the theory. The important work of Froissart has shown that arbitrariness from subtractions cannot be present above the p wave.[3] The analogy with nonrelativistic problems has encouraged the hope that relativistic amplitudes are also completely determined by the "potentials" D_t and D_u.

This hope has sometimes been stated as a postulate of S-matrix theory. Such a postulate has recently been given an elegant formulation within the framework of the ideas of complex angular momentum. Chew[2,4] has called this principle maximal analyticity of the second degree (MASD). MASD requires that the amplitude for all values of angular momentum be connected by analytic continuation. At the present stage, MASD enters S-matrix theory as a postulate, although subsequent developments may show that it follows from other postulated properties of the S matrix.

It is perhaps worth pointing out the exact relationship of the present paper to the work of Froissart and Martin.[2] These authors examine the question of subtractions by studying the limitations placed on the scattering amplitude by unitarity, analyticity, and crossing symmetry. They are able to conclude that relatively few (sometimes none) of the subtraction terms can be arbitrary. Subtractions generally have to be made, but they are determined by the spectral functions. In these discussions polynomial bounds on the scattering amplitude are assumed, but no assumption is made about the explicit form of the asymptotic behavior. Although most (or all) of the subtraction terms are, in principle, determined, it is not generally possible to write down an explicit formula for these subtraction functions.

In the discussion which follows, we take a different viewpoint from the one just outlined. We assume an explicit form for the asymptotic behavior of the amplitude, suggested by certain types of singularities in the complex angular momentum plane. Unitarity is never used, but the strong assumption of the exact form of the asymptotic behavior generally implies explicit formulas for the subtraction functions (Sec. II) and also leads to a demonstration that physical partial-wave amplitudes at lower angular momenta are linked by analytic continuation to physical partial waves at high angular momenta (Sec. III).

Special mention is also made here of a point first emphasized by Chew and Jones,[5] that it is not necessary for the singularities in the angular-momentum plane to retreat to the left of the imaginary axis in order for all subtraction terms to be fully determined. We also show that cuts in the angular-momentum plane do not appear to raise any special difficulties in connection with the determination of subtractions.

* National Science Foundation Postdoctoral Fellow, 1964.
[1] M. Jacob and G. F. Chew, *Strong Interaction Physics* (W. A. Benjamin, Inc., New York, 1964). For another good discussion of the question of subtractions, see the articles by A. Martin and R. Oehme in *Strong Interactions and High Energy Physics* (Oliver and Boyd, Edinburgh and London, 1964).
[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).
[3] M. Froissart, Phys. Rev. **123**, 1053 (1961). See also A. Martin, *ibid.* **129**, 1432 (1963), who achieves stronger conclusions than Froissart in special cases.
[4] G. F. Chew, Lawrence Radiation Laboratory Report UCRL-10786 (unpublished).
[5] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

II. DETERMINATION OF SUBTRACTION TERMS

We consider here several types of asymptotic behavior which may be experienced by the full scattering amplitude. By giving the form of the asymptotic behavior we are immediately led, in all but one of the examples, to formulas giving the complete determination of the subtraction terms by the absorptive parts. The one example we consider where the subtraction terms are not completely determined is the well-known case of polynomial-type behavior.

Although our development in this section proceeds without explicit use of the complex angular-momentum plane, it will be recognized that the forms of asymptotic behavior we consider are the same as would be produced by certain types of singularities in the angular-momentum variable.

Our starting point is a fixed energy dispersion relation for the invariant scattering amplitude:

$$\begin{aligned} A(s,t) &= \frac{t^N}{\pi}\int_{t_0}^{\infty} dt' \frac{D_t(t',s)}{(t')^N(t'-t)} + \sum_{i=0}^{N-1} b_i(s)t^i , \\ s &= 4(\nu+1) , \\ t &= -2\nu(1-\cos\theta) . \end{aligned} \tag{II.1}$$

Our notation and conventions are as follows: ν is the center-of-mass three-momentum squared; θ is the scattering angle in the center-of-mass system; D_t is the absorptive part in the t channel; the b_i are subtraction terms; t_0 is threshold; we consider here elastic scattering of two spinless particles of unit mass. We have omitted terms that arise from the absorptive part in the u channel because such terms in no way modify the arguments which are to follow. It is to be noted that Eq. (II.1) is written for fixed s, and if Mandelstam cuts[6] enter the problem, there will be no finite value of N which makes the dispersion integrals converge for all values of s.

As a first example of the possible asymptotic behavior of $A(s,t)$, let us suppose that there exists a small neighborhood of s, where

$$A(s,t) \underset{t\to\infty}{\longrightarrow} 0 . \tag{II.2}$$

This just means, of course, that the subtractions made in (II.1) are unnecessary and that the $b_i(s)$ are related to $D_t(t,s)$ by

$$b_i(s) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{D_t(t',s)}{(t')^{i+1}} . \tag{II.3}$$

What has happened here, of course, is that the polynomial in t in Eq. (II.1) had to be canceled by the integral term, which requirement leads to Eq. (II.3). Although the integral in (II.3) will, in general, only be defined over a neighbor of s values, still the function $b_i(s)$ so defined is an analytic function of s and can, in principle, be continued to all s values (see, in this regard, Ref. 3). Thus the subtraction terms $b_i(s)$, in this case, are completely determined by D_t. From the point of view of complex angular-momentum plane, this example corresponds to the case where all singularities have retreated to the left half plane.

We consider another example where the subtraction terms in (II.1) are completely determined. Suppose over an interval of s that $A(s,t)$ has the following high t behavior:

$$\begin{aligned} A(s,t) &\underset{t\to\infty}{\longrightarrow} \sum_{i=1}^{M} C_i(s) t^{\alpha_i(s)} , \\ D_t(t,s) &\underset{t\to\infty}{\longrightarrow} \sum_{i=1}^{M} \tilde{C}_i(s) t^{\alpha_i(s)} . \end{aligned} \tag{II.4}$$

It is understood that the terms neglected in (II.4) vanish at high t. The only condition we impose on the $\alpha_i(s)$ is that for some range of s values the $\alpha_i(s)$ not be integers. For the remainder of our discussion, we shall, for simplicity, consider just one asymptotic term in (II.4), which we designate $C(s)t^{\alpha(s)}$ without subscripts. The generalization to any finite number of similar terms will be obvious. We then set

$$D_t(t,s) = \tilde{C}(s) t^{\alpha(s)} + \tilde{D}_t(t,s) , \tag{II.5}$$

where

$$\tilde{D}_t(t,s) \underset{t\to\infty}{\longrightarrow} 0 .$$

In writing Eq. (II.1), subtractions are clearly required if $\mathrm{Re}\,\alpha(s) > 0$. The integer N is to be selected so that

$$\mathrm{Re}\,\alpha + 1 > N > \mathrm{Re}\,\alpha .$$

Thus we have

$$A(s,t) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{\tilde{D}_t(t',s)}{t'-t} + \frac{t^N}{\pi} \times \int_{t_0}^{\infty} dt' \frac{\tilde{C}(s)(t')^{\alpha(s)-N}}{t'-t} + \sum_{i=0}^{N-1} b_i(s) t^i . \tag{II.6}$$

All integrals in (II.6) converge. Now we simply observe that

$$\frac{t^N}{\pi}\int_{t_0}^{\infty} dt' \frac{(t')^{\alpha-N}}{t'-t} = -\frac{e^{-i\pi\alpha} t^{\alpha}}{\sin\pi\alpha} - \frac{t^N}{\pi}\int_{0}^{t_0} dt' \frac{(t')^{\alpha-N}}{t'-t} . \tag{II.7}$$

Requiring a cancellation of the asymptotic integral powers of t leads at once to a determination of the $b_i(s)$:

$$b_i(s) = -\frac{\tilde{C}(s)}{\pi}\int_{0}^{t_0} dt' (t')^{\alpha(s)-1-i} . \tag{II.8}$$

If we assume that $\tilde{C}(s)$ and $\alpha(s)$ are analytic functions, then, as before, the amplitude becomes determined by

[6] S. Mandelstam, Nuovo Cimento **30**, 1113, 1127, and 1148 (1963).

D_t over a range of s values, and hence it is known everywhere by analytic continuation. This example, of course, corresponds to simple poles in the right half angular-momentum plane.

This last example may be compared with a discussion by Chew, Frautschi, and Mandelstam,[7] where they indicate how an amplitude with the same type Regge-pole behavior may be completely determined by the absorptive part. They, however, in order to define integrals without introducing subtraction terms, assume that $\mathrm{Re}\alpha<0$ for some values of s. This makes the case they consider really correspond to the first example given in this section. We note that in our last example no assumption was made that $\mathrm{Re}\alpha$ be less than zero for any s.

As a further example, we consider an integral superposition of power behaviors producing the asymptotic behavior

$$A(s,t) \underset{t\to\infty}{\to} C(s)\int_A^B d\alpha\, g(\alpha)t^\alpha + \text{vanishing terms}\,,$$
$$D_t(t,s) \underset{t\to\infty}{\to} \tilde{C}(s)\int_A^B d\alpha\, \tilde{g}(\alpha)t^\alpha\,, \qquad \text{(II.9)}$$

where $\tilde{g}(\alpha)$ is assumed to be real over the contour from A to B.

Again we can determine the subtraction terms $b_i(s)$ by the general procedure following Eq. (II.6). In this, case the equation comparable to (II.7) is

$$\frac{t^N}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\int_A^B d\alpha\,\tilde{g}(\alpha)(t')^{\alpha-N}$$
$$=-\frac{e^{-i\pi\alpha}}{\sin\pi\alpha}\int_A^B d\alpha\,\tilde{g}(\alpha)t^\alpha$$
$$-\frac{t^N}{\pi}\int_0^{t_0}\frac{dt'}{t'-t}\int_A^B d\alpha\,\tilde{g}(\alpha)t^\alpha. \qquad \text{(II.10)}$$

Thus we have

$$b_i(s)=-\frac{\tilde{C}(s)}{\pi}\int_0^{t_0}dt'\int_A^B d\alpha\, g(\alpha)(t')^{\alpha-1-i}. \qquad \text{(II.11)}$$

This last example is a type of asymptotic behavior corresponding to a cut in the angular-momentum plane with branchpoints at $l=A$ and $l=B$ (see, in this connection, Ref. 8). As mentioned earlier, the cuts proposed by Mandelstam[6] do not restrict their movements to the left of some given vertical line in the angular-momentum plane for all values of s in the physical sheet. This means that, generally, a finite number of subtractions will not suffice for all values of s. It is interesting to note that this state of affairs has not posed a problem here, for we only need assume the validity of dispersion relations (II.1) over a small neighborhood of s values in order to define the amplitude completely.

From these examples, it is easy to conclude that the possibility of a complete determination of the subtraction terms $b_i(s)$ in terms of the absorptive part $D_t(t,s)$ is closely connected to the asymptotic behavior of the amplitude $A(s,t)$ in momentum transfer t. It is important to demonstrate the failure of D_t to determine the $b_i(s)$ under still another type of asymptotic t behavior. For this point, it will be convenient to rewrite the subtraction terms using Legendre polynomials as follows:

$$\sum_{i=0}^{N-1} b_i(s)t^i=\sum_{l=0}^{N-1}(2l+1)a_l(s)P_l\left(1+\frac{t}{2\nu}\right). \qquad \text{(II.12)}$$

Suppose that we assume $A(s,t)$ has a term in its asymptotic behavior proportional to $P_{l_0}(1+t/2\nu)$, where l_0 is a positive integer or zero. That is to say,

$$A(s,t) \xrightarrow[s\text{ fixed},\ t\to\infty]{} \text{const}[1+(t/2\nu)]^{l_0} + \text{other terms}. \qquad \text{(II.13)}$$

The important part of the assumption is that the "other terms" do not cancel the first term in (II.13). Now it is clear that our previous arguments cannot be repeated. The cancellation mechanism, which operated before between the polynomial subtraction terms and the integral terms over D_t, is no longer in force, and l_0th partial wave cannot be determined from $D_t(t,s)$.

To summarize, we have seen that certain assumptions about the asymptotic behavior of the amplitude in the momentum-transfer t lead to a complete determination of all subtraction terms $b_i(s)$ in terms of the absorptive part $D_t(t,s)$. However, if the asymptotic behavior of the amplitude in the t variable is polynomial type, the determination is not possible.[9] Although our discussion has been based upon a few isolated examples, the following generalization seems fairly obvious: If the asymptotic behavior of the amplitude in t is *bounded by* but not equal to polynomial behavior for some neighborhood of s values, then certainly, for a very wide class of such behaviors, the functions $b_i(s)$ become completely determined in terms of $D_t(t,s)$. We emphasize several points in connection with our results: (1) For a complete determination of the $b_i(s)$, it is not necessary to require that $D_t(t,s)$ vanish at high t for some values of s as was assumed in our first example. (2) In order to give an

[7] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962). See also R. Blankenbecler and M. L. Goldberger *ibid.* **126**, 766 (1962).

[8] M. Froissart, High Energy Properties of the Mandelstam Representation, Seminar on Theoretical Physics, (International Atomic Energy Commission, Trieste, 1962).

[9] It was pointed out to the author by Professor K. A. Johnson that asymptotic t behavior involving an essential singularity at infinity (such as e^{-t}) is obviously another type in which subtraction terms cannot be determined or even an ordinary dispersion relation written for the amplitude. There seem to be, however, no suggestions at present that such a behavior might be physically realized.

explicit formula for the determination of the $b_i(s)$ such as (II.8), it is necessary to know the explicit form of the asymptotic behavior.

III. MAXIMAL ANALYTICITY OF THE SECOND DEGREE

In this section, we illustrate the link between our preceding discussion and MASD. As an example, we assume asymptotic t behavior of the form (II.4) and (II.5). Our goal is to show that if the asymptotic behavior is of this form, then the low-lying angular momentum amplitudes are determined by analytic continuation from high-angular momenta. On the other hand, if the asymptotic behavior includes polynomial terms of the type (II.12), then the actual and interpolated partial-wave amplitudes for $l=l_0$ will not be the same. That an agreement of the actual and interpolated partial-wave amplitudes implies the absence of asymptotic terms of the form (II.13) has been previously demonstrated.[1] We give here a demonstration of the converse statement: namely, the absence of polynomial terms implying agreement of the actual and interpolated amplitudes. That the actual physical partial-wave amplitude should agree with the amplitude analytically continued from a region of high Rel is the statement of MASD.

Our starting point is Eq. (II.6). We define the partial-wave amplitude in the usual way according to

$$A(s,l)=\frac{1}{2}\int_{-1}^{+1} dz\, P_l(z) A[s,\, -2\nu(1-z)]. \qquad \text{(III.1)}$$

If $l>N$, then it is easy to deduce from Eq. (II.6) that

$$A(s,l)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{t'}{2\nu}Q_l\left(1+\frac{t'}{2\nu}\right)\tilde{D}_l(t',s) +\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{2\nu}Q_l\left(1+\frac{t'}{2\nu}\right)\tilde{C}(s)(t')^{\alpha(s)}, \qquad \mathrm{Re}\,l>N, \qquad \text{(III.2)}$$

where Q_l is a Legendre function of the second kind. The asymptotic behavior of D_t is such that the first integral in (III.2) converges for all physical l values, so we shall often omit it in the equations that follow. It will be recognized that Eq. (III.2) just represents the well-known unique Froissart-Gribov[10] amplitude, which coincides with the physical partial-wave amplitudes for $l>N$ and is a holomorphic function of l for Re$l>N$ (some range of s values is implied by this statement).

Unless there are natural boundaries in l, the amplitude in (III.2) can be continued to values with Re$l\leq N$. One, of course, cannot make the continuation directly via Eq. (III.2), because the integral over $t^{\alpha(s)}$ will cease to be defined. In what follows, we shall give explicit formulas for doing the continuation.

We now do the projection (III.1) in the case with $l=l_0$, where l_0 is a physical value (positive integer) less than N. Utilizing Eqs. (II.6) and (II.12) and omitting the term which includes $\tilde{D}_t$, we arrive, after some manipulation, at the result

$$A(s,l_0)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{2\nu}\tilde{C}(s)(t')^{\alpha(s)} \times\left[Q_{l_0}\left(1+\frac{t'}{2\nu}\right)-\sum_{j=l_0+1}^{j=N}\frac{d_j(l_0,s)}{(t')^j}\right] +a_l(s)\delta_{ll_0}, \quad l_0<N. \qquad \text{(III.3)}$$

where the $d_j(l_0,s)$ are coefficients of an asymptotic expansion of $Q_l(1+t'/2\nu)$ in t' and δ_{ll_0} is the ordinary Kronecker delta. It will be seen that the integral in Eq. (III.3) converges, while, since $l_0<N$, the integral in (III.2) does not converge. The convergence of terms appearing in Eq. (III.3) was assured, for (III.3) was derived from (III.1) which is clearly convergent and well defined.

The next step is to begin with Eq. (III.2), which is defined for Re$l>N$, and analytically continue in l down to the point $l=l_0$, so that the result of this continuation procedure may be compared with (III.3). In order to accomplish the continuation of (III.2), we use a trick first employed by Barut and Zwanziger.[11] Since we are only interested in the continuation as far down as $l=l_0$, we write[11]

$$Q_l\left(1+\frac{t'}{2\nu}\right)=\left[Q_l\left(1+\frac{t'}{2\nu}\right)-\frac{1}{(t')^{l+1}}\sum_{j=0}^{j=N-l_0-1}\frac{d_j(l,s)}{(t')^j}\right] +\frac{1}{(t')^{l+1}}\sum_{j=0}^{j=N-l_0-1}\frac{\tilde{d}_j(l,s)}{(t')^j}. \qquad \text{(III.4)}$$

The term of (III.4) enclosed in square brackets is so constructed that the asymptotic powers of Q_l in t' are cancelled in such a way that, when $l=l_0$, the bracketed expression behaves as $1/(t')^{N+1}$ for large t'; for Re$l>l_0$, the expression is even more convergent. The reader will notice a similarity between the bracketed expressions in Eqs. (III.3) and (III.4); in fact, the two expressions are identical when $l=l_0$, a fact which will be exploited in what follows. However, beyond this the expressions are quite different in their general properties. For example, the bracketed expression in (III.4) is an analytic function of l, while the corresponding expression in (III.3) is *not* an analytic function of l_0 because l_0 appears as a discrete index in the latter.

We now insert (III.4) into (III.2) to obtain (omitting,

[10] M. Froissart, Report to the La Jolla Conference on Theoretical Physics, June 1961 (unpublished) and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 and 1962 (1961). [English transl.: Soviet Phys.—JETP **14**, 478 and 1395 (1962)].

[11] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).

as before, the D_t term):

$$A(s,l)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{2\nu}\left[Q_l\left(1+\frac{t'}{2\nu}\right)-\frac{1}{(t')^{l+1}}\sum_{j=0}^{j=N-l_0-1}\frac{\tilde{d}_j(l,s)}{(t')^j}\right]\times\tilde{C}(s)(t')^{\alpha(s)}+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{2\nu}\frac{1}{(t')^{l+1}}\times\sum_{j=0}^{j=N-l_0-1}\frac{\tilde{d}_j(l,s)}{(t')^j}\tilde{C}(s)t^{\alpha(s)}. \quad \text{(III.5)}$$

Now the game is simple: The first integral term in (III.5) converges and hence is a holomorphic function for all $l\geq l_0$. The second integral in (III.5) may now be explicitly integrated for $\mathrm{Re}\,l>N$ where it converges, and the result may then be continued to all points in l. We have the result that the second integral in (III.5) is equal to the following expression:

$$-\sum_{j=0}^{j=N-l_0-1}\frac{\tilde{d}_j(l,s)}{2\pi\nu}\frac{\tilde{C}(s)t_0{}^{\alpha-l-j}}{\alpha-l-j}. \quad \text{(III.6)}$$

So, finally, setting $l=l_0$ in (III.5) and doing some juggling over the summation indices, we obtain:

$$\tilde{A}(s,l_0)=\frac{1}{\pi}\int_{t_0}\frac{dt'}{2\nu}\left[Q_{l_0}\left(1+\frac{t'}{2\nu}\right)-\sum_{k=l_0+1}^{k=N}\frac{\tilde{d}_{k-l_0-1}(l_0,s)}{(t')^k}\right]\times\tilde{C}(s)(t')^{\alpha(s)}-\sum_{j=0}^{j=N-l_0-1}\frac{\tilde{d}_j(l,s)}{2\pi\nu}\times\frac{\tilde{C}(s)t_0{}^{\alpha-l_0-j}}{\alpha-l_0-j}, \quad \text{(III.7)}$$

where the tilde over the A is a reminder that this determination of the amplitude for $l=l_0$ was achieved differently from that of (III.3). We now proceed to show that $\tilde{A}(s,l_0)=A(s,l_0)$. Comparing Eqs. (III.3) and (III.4) and recalling the definitions of $d_j(s)$ and $\tilde{d}_j(s)$, we see by inspection that

$$d_j(l_0,s)=\tilde{d}_{j-l_0-1}(l_0,s),$$

and thus we have agreement of the integral terms in (III.3) and (III.7). To compare the remaining terms, we recall that $a_{l_0}(s)$ in Eq. (III.3) is given by

$$a_{l_0}(s)=\frac{1}{2}\int_{-1}^{+1}dz\,P_{l_0}(z)\sum_{i=0}^{N-1}b_i(s)[t(z)]^i$$

$$=\frac{1}{2}\int_{-4\nu}^{0}\frac{dt}{2\nu}P_{l_0}\left(1+\frac{t}{2\nu}\right)\sum_{i=0}^{N-1}b_i(s)t^i, \quad \text{(III.8)}$$

where from (II.8) we have

$$b_i(s)=-\frac{\tilde{C}(s)}{\pi}\int_0^{t_0}dt'(t')^{\alpha(s)-1-i}$$

$$=\frac{-\tilde{C}(s)}{\pi}\frac{(t_0)^{\alpha(s)-i}}{\alpha(s)-i}. \quad \text{(III.9)}$$

Thus, all that remains is an examination of the quantities

$$\frac{1}{2}\int_{-4\nu}^{0}dt\,P_{l_0}\left(1+\frac{t}{2\nu}\right)t^i. \quad \text{(III.10)}$$

First it is clear that the quantity (III.10) due to orthogonality properties of Legendre polynomials will vanish for $i<l_0$. For $i>l_0$, we use the identity

$$Q_{l_0}(z)=-\frac{1}{2}\int_{-1}^{+1}\frac{dz'}{z'-z}P_{l_0}(z')$$

to infer

$$\frac{1}{2}\int_{-4\nu}^{0}dt\,P_{l_0}\left(1+\frac{t}{2\nu}\right)t^{i+l_0}=\tilde{d}_i(l_0,s),\quad i=0,1,2,\cdots. \quad \text{(III.11)}$$

Combining Eqs. (III.8)–(III.11) we arrive at the result

$$a_{l_0}(s)=-\sum_{i=0}^{i=N-1-l_0}\frac{\tilde{C}(s)}{2\pi\nu}\frac{t_0{}^{\alpha(s)-i-l_0}}{\alpha(s)-i-l_0}\tilde{d}_i(l_0,s), \quad \text{(III.12)}$$

which provides the complete agreement between Eqs. (III.7) and (III.3) and proves that

$$\tilde{A}(s,l_0)=A(s,l_0).$$

Thus, we have proved that the amplitude obtained by analytic continuation from high l and the amplitude obtained by a straightforward projection agree at $l=l_0$, where $l_0<N$ under the assumption that the asymptotic behavior of the amplitude in t is of the form $t^{\alpha(s)}$. This type of behavior will be recognized as that which occurs when Regge poles only are present in the right half l plane. We do not, however, wish to imply that this is the only behavior that will result in an agreement between the interpolated and physical amplitudes. In fact, our result will probably hold if there are cuts in the l plane[6] or any other singularity structure, with the exception of a natural boundary which probably cannot occur if the amplitude is bounded by a polynomial. We have, of course, only given the explicit demonstration for the simplest situation, that of Regge-pole behavior.

Our result, that the physical and interpolated partial-wave amplitudes agree under the assumption of a certain type of asymptotic behavior, was perhaps to be expected. However, the result is not entirely empty. In a different line of argument in which the explicit form of the asymptotic behavior is not assumed but unitarity and polynomial boundedness are (as discussed in the introduction), Martin[1] has shown that, in some cases, all partial-wave amplitudes become uniquely determined by the spectral functions. However, even in these cases he is not able to conclude that the s- and p-wave physical amplitudes agree with the partial-wave amplitude analytically continued from high l. Our example, under the assumption of an explicit form for the high momentum-transfer has been explicitly shown to have this property.

As a final remark, we mention the fact[1] that if the asymptotic behavior of the amplitude is of polynomial type, then it would be impossible to have agreement between the interpolated and the physical partial-wave amplitude. It is clear that terms of polynomial type introduce Kronecker deltas into the amplitude which give contributions at discrete values of angular momentum, which contributions cannot be reproduced by a smooth connection with higher angular momenta. This gives the important result[1] that MASD forbids polynomial asymptotic behavior of the scattering amplitude in t.

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Bounds for a Class of Bethe-Salpeter Amplitudes*

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For a certain wide class of kernels involving trilinear coupling of scalar particles, the absorptive part of the Bethe-Salpeter amplitude for forward scattering is bounded from above and below. The bounds are expressed in the form $B_1 s^{\alpha_1} \leqslant A(s) \leqslant B_2 S^{\alpha_2}$, where s is the squared c.m. energy and B_1 and B_2 are positive constants. Expressions for the exponents α_1 and α_2 are given as functions of the coupling constant g. For the straight ladder model, α_1 and α_2 coincide for all values of g, the common expression agreeing with an exact result of Nakanishi. For the more complicated models, α_1 and α_2 do not in general coincide. However, in the strong-coupling limit $g \to \infty$, we find that $\alpha_2/\alpha_1 \to 1$; moreover, the common asymptotic behavior $\alpha_{1,2} \to_{g\to\infty} g/4\pi m$ is the same for all the models, including the straight-ladder model.

I. INTRODUCTION

USING techniques discussed in two earlier papers,[1,2] we consider here the problem of setting upper and lower bounds on the absorptive part of the forward elastic scattering amplitude for a certain wide class of ladder-like models. We deal with theories involving scalar particles which couple trilinearly.

In general, the absorptive amplitude A satisfies a Bethe-Salpeter equation, as symbolized in Fig. 1. For an inclusive treatment, one would have to take for the kernel K a sum over all possible irreducible diagrams; and for the Born term A_B a similar sum, evaluated on the mass shell $k^2=0$. But as we shall understand the term here, a particular *model* is characterized by the choice of a particular one of the irreducible diagrams for the kernel K and corresponding Born term A_B. The class of such models which will come under discussion here is characterized by the examples shown in Fig. 2 for the irreducible kernels. The heavy lines (spinless "nucleons") correspond to particles of mass m, except for the *external* nucleons which are taken, for reasons of kinematic simplicity, to be massless. The wavy lines represent exchanged particles ("mesons"). In general terms, the class of irreducible diagrams which we consider consists of those in which each wavy line joins two solid lines, without further connections (no loops or self-energy and vertex corrections).

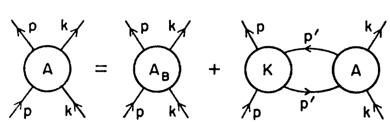

FIG. 1. Notation for the integral equation.

Insofar as the kernel K is concerned, the exchanged particles are taken to be massless. But in the Born term A_B, which is described by the same diagram as for K, we suppose that *one* of the exchanged particles has a finite mass μ. Although we could set $\mu=0$ without embarrassment insofar as the absorptive amplitude is concerned, we would encounter infrared divergence troubles for the real part of the amplitude. In order to

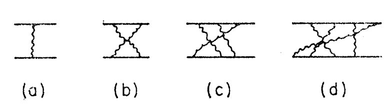

FIG. 2. Examples of irreducible diagrams.

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[1] G. Tiktopoulos and S. B. Treiman, Phys. Rev. **135**, B711 (1964).

[2] G. Tiktopoulos and S. B. Treiman, Phys. Rev. **136**, B1217 (1964).